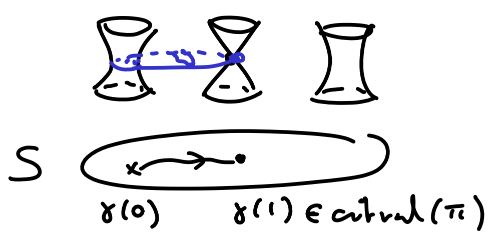
13/10/11 - Lefschetz fibrations cont'd: monodromy & Dehn twists ①

Recall: • Lefschetz fibration = $\pi : (E, \omega, J) \longrightarrow (S, j)$
exact sympl. with corners — Riem. surface w/ boundary

st
- (j, J)-holom.
- submersion away from isolated crit pts ; [crit. values distinct]
- local model at crit. pt: $Q : \mathbb{C}^{n+1} \to \mathbb{C}$, $(z_1 \dots z_{n+1}) \longmapsto \sum z_i^2$
- convexity, horizontality conditions at boundary

• Parallel transport: $TE^h = (\ker d\pi)^{\perp \omega}$ horiz. distrib^n (outside crit π)
Parallel transport induces exact symplecto b/w fibers.

• Vanishing paths, vanishing cycles, thimbles:

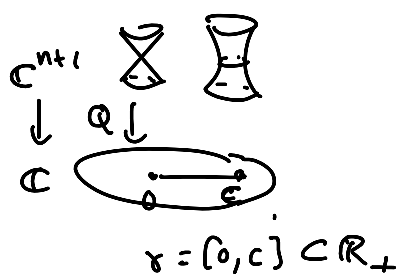


$\gamma : [0,1] \to S$, $1 \longmapsto$ crit. value

$\Rightarrow$ vanishing cycle:
$V_\gamma \subset E_{\gamma(0)}$ exact Lag. sphere
= pts whose parallel transport $\to$ crit pt

$\Rightarrow$ thimble: $\Delta_\gamma \subset E$ Lag. ball, $\partial \Delta_\gamma = V_\gamma$
(= union of par. transport of V_γ along γ)

• Local model.



Smooth fibers $\{ \sum z_i^2 = c \} \cong T^*S^n$

$\Rightarrow V_\gamma \simeq$ zero section in T^*S^n
$= \{ \mathrm{Im}\, z = 0,\ Q(z) = c \}$
$= \{ (x_1 \dots x_{n+1}) \in \mathbb{R}^{n+1} / \sum x_i^2 = c \}$

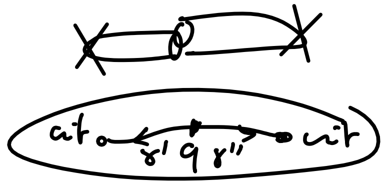
and $\Delta_\gamma = \{ (x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} / |x|^2 \leq c \}$.

This is a natural source of Lag's!

Lemma: γ embedded path $\subset S \setminus \mathrm{critv}(\pi)$, $L \subset E$ Lag. st. $\pi(L) = \gamma$
$\Rightarrow L \cap E_{\gamma(t)}$ Lag. in $E_{\gamma(t)}$, and L obtained from it by parallel transport along γ.

(Pf: $\Lambda = L \cap E_{\gamma(t)}$ dim. n, isotropic, but then remaining dir^n $v \in (T\Lambda)^\omega = T\Lambda \oplus TE^h$)
for generic t (regular val. of $\pi_{|L}$)

- Matching cycles:



$\gamma = \gamma' \cup \gamma''$ embedded, γ', γ'' vanishing paths
$\gamma'(0) = \gamma''(0) = q$

$\Rightarrow V_{\gamma'}, V_{\gamma''} \subset E_q$ Lag. spheres

* If $V_{\gamma'} = V_{\gamma''}$ then $\Sigma_\gamma := \Delta_{\gamma'} \cup \Delta_{\gamma''}$ smooth Lag. $S^{n+1} \subset E$ fibers over γ

* More generally, if $V_{\gamma'} \underset{\text{Ham. iso.}}{\simeq} V_{\gamma''}$ in E_q, then can modify ω_E by an exact deformation to make them match $\rightsquigarrow \Sigma_\gamma \subset E$

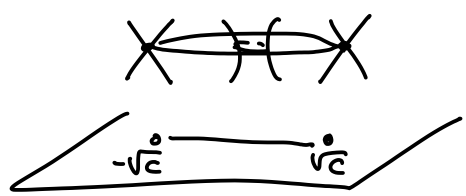
Or, by Moser, deforming $\omega_E \Leftrightarrow$ deforming π to get matching.

Call Σ_γ a matching cycle
γ —— " —— path

Ex: $E = \left\{ (z_1 \ldots z_{n+1}) \in \mathbb{C}^{n+1} \,\middle|\, \sum_1^{n+1} z_i^2 = c \right\} \xrightarrow{p} \mathbb{C}$ (or truncation)
$(\simeq T^*S^n)$ $\qquad (z_1 \ldots z_{n+1}) \longmapsto z_{n+1}$

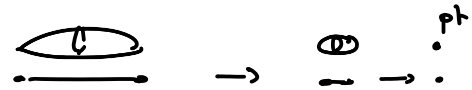
Lefschetz fibration: fiber $p^{-1}(y) = \left\{ (z_1 \ldots z_n) \in \mathbb{C}^n \,/\, \sum_1^n z_i^2 = c - y^2 \right\}$
$\simeq T^*S^{n-1}$!

2 sing fibers: at $y = \pm\sqrt{c}$.



The segment $[-\sqrt{c}, \sqrt{c}]$ is a matching path,
matching cycle = the sphere $\sqrt{c}\, S^n \subseteq E$!

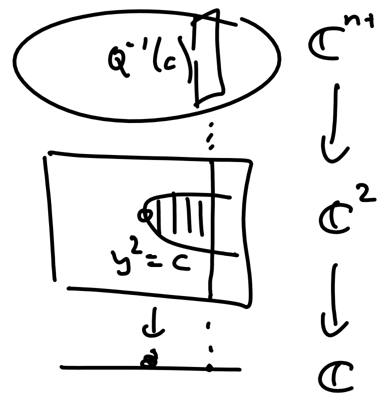
What happens as vary c? For $c \to 0$,



See higher dim local model from lower dim. this way!

NB: we've viewed each fiber of $Q : \mathbb{C}^{n+1} \to \mathbb{C}$ as *itself* a Lefschetz fibration wrt $p : Q^{-1}(c) \to \mathbb{C}$:

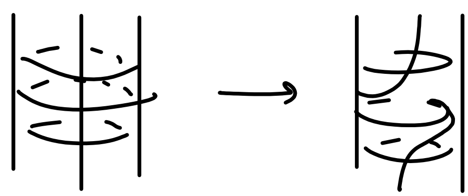
Seidel calls this a "bi-fibration"
& it's a natural source of matching paths



- **Dehn twists:** $V \subset (M, \omega = d\theta)$ exact Lag. sphere
(parametrized, ie. fix $S^n \simeq V$)

$\rightsquigarrow \tau_V \in Symp(M)$ exact symplecto, supp^d in nbd(V)
(canonical up to Ham. isotopy)

In dim 1:



rotate each S^1 by varying amount $0 \nearrow 2\pi$.

Observe: this is rescaled geodesic flow for S^1 w/ standard metric

$TS^n \simeq T^*S^n$ w/ round metric, geodesic flow:

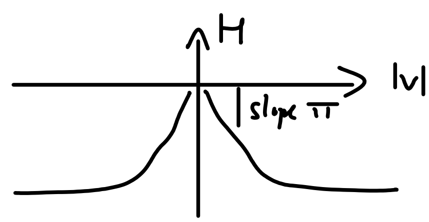
x, $v \in T_x S^n$
$\Rightarrow$ follow geodesic in dir. of v.

It's Hamiltonian! ... kind of

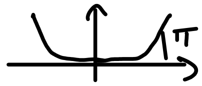
$H(x,v) = h(|v|)$ gives: parallel transport by distance $h'(|v|)$.

eg. $|v|^2$ gives // transport by distance $|v|$.

So: take $H = h(|v|)$,



This gives // transport by amounts $\nearrow$ to π as approach 0 section & induces antipodal map on zero section.

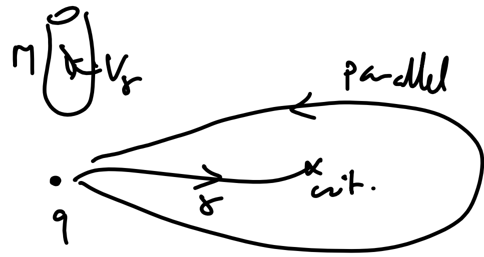
Or: (better) take $H =$ 

& compose with antipodal $(x,v) \to (-x,-v)$.

Remark: in $\dim_{\mathbb{C}} 2$, eg on T^*S^2, τ_V^2 is isotopic to Id in Diff(M) [classical]

but in general not in Symp(M) [Seidel: $HF(L_0, \tau_V^2(L_1)) \neq HF(L_0, L_1)$].

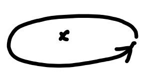
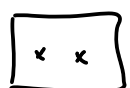
Prop: Monodromy of L. fibration around a crit. value is Ham. isotopic to τ_V where V = vanishing cycle



parallel transport / λ induces $h_\lambda \in \mathrm{Symp}(M, \partial M)$ $(M = \pi^{-1}(q))$.

$h_\lambda \underset{\text{Ham. iso.}}{\simeq} \tau_V$

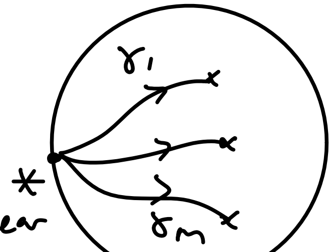
(Can see it by explicit computation on local model, or from bifibration picture:

 $Q^{-1}(c) \xrightarrow[\text{proj. to } z_{n+1}]{}$  crit $= \pm\sqrt{c}$

As c goes around origin,  $\pm\sqrt{c}$ ...)

- **Def:** a distinguished basis of vanishing paths $\gamma = (\gamma_1, \dots, \gamma_m)$

=



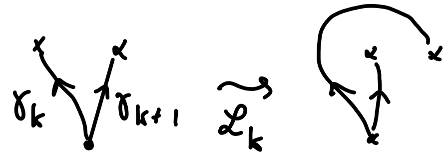
$*$ at or near boundary

$S = D^2$

reaching all critvals (π) in clockwise order.

To this associate ordered sequence of v.c.'s $(V_1, \dots, V_m) \subset M = \pi^{-1}(*)$

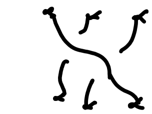
- Mutations: when $S = D^2$, any two bases of vanishing paths are related (up to isotopy) by "Hurwitz moves"



& inverse move

induce $(V_1 \dots, V_k, V_{k+1} \dots V_m) \overset{\mathcal{L}_k}{\longmapsto} (V_1, \dots, \tau_{V_k}(V_{k+1}), V_k, \dots V_m)$

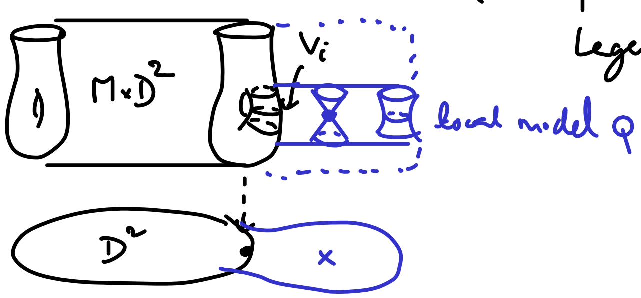
(converse: $(\dots V_k, V_{k+1}, \dots) \overset{\mathcal{L}_k^{-1}}{\longmapsto} (\dots, V_{k+1}, \tau_{V_{k+1}}^{-1}(V_k) \dots)$)

NB: These induce an action of the braid group $B_m = \pi_0 \mathrm{Diff}^+(D^2, m \text{ pts})$ 

Prop: The Hurwitz equiv^{ce} class of $(V_1 \dots V_m)$ is an inv^t of the Lefschetz fibration over D^2. Conversely, given any collection of exact Lag. spheres $(V_1 \dots V_m)$ in exact sympl. mfld M, can build a L. fibration over D^2 with vanishing cycles $V_1 \dots V_m$, unique up to exact symplectic deformation.

Construction: start with $M \times D^2$, glue local models near copy of V_i in fibers at boundary, and enlarge/round corners.

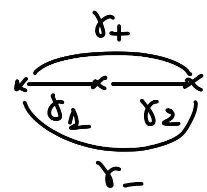
($\leftrightarrow$ top: attach a standard "Weinstein" handle along Legendrian sphere $V_i \times \{pt\} \subset \partial(M \times D^2)$)



* To build "exotic" sympl. mflds: start w/ M containing many interesting Lagrangian spheres, incl. some that are smoothly isotopic but not Ham. iso., and custom-build L. fibrations w/ fiber M.

Typically, this is done using M = itself carrying a L. fibration:

$$M \xrightarrow[p]{} D^2, \quad \text{Lag. spheres = matching spheres for } p.$$

Ex: if all paths match, 

$$\Sigma_{\gamma_+} \simeq \tau_{\Sigma_{\gamma_1}}(\Sigma_{\gamma_2})$$
$$\Sigma_{\gamma_-} \simeq \tau^{-1}_{\Sigma_{\gamma_1}}(\Sigma_{\gamma_2})$$

so eg. if $\dim_{\mathbb{R}} M = 4$, we get $\Sigma_{\gamma_+} \underset{C^\infty \text{iso.}}{\simeq} \Sigma_{\gamma_-}$

but not necess. Lag. iso!